Solution to Problem Set 11
Optical Waveguides and Fibers (OWF)

Exercise 1: Dispersion compensation in optical fiber networks

In the early 90s, fiber links were installed between major cities in Germany. At that time, only standard single-mode fibers (SMF) with $C_\lambda = 16\,\text{ps}/(\text{nm km})$ at $\lambda_0 = 1.55\,\mu\text{m}$ were used. For high data rates and long transmission distances, this dispersion is too high, therefore it is necessary to install dispersion-compensating fibers (DCF) with negative dispersion after regular distances. In this way, one obtains a system of the type SMF-DCF-SMF-DCF... .

a) Assume that a DCF with $C_\lambda = -76\,\text{ps}/(\text{nm km})$ should be installed after 100 km of SMF. How long should this DCF be?

Solution:

$$L_{\text{DCF}} = -\frac{C_{\lambda,\text{SMF}}}{C_{\lambda,\text{DCF}}} L_{\text{SMF}} = 21.05\,\text{km}.$$

b) Consider a Gaussian and chirp-free pulse with a temporal width at half of the peak power (FWHM) of 50 ps at a center wavelength of 1550 nm. This pulse is launched into two different fibers, an SMF of 100 km length and a DCF having the length calculated in part a). Calculate the pulse duration at the output of both fibers. How do the two pulses differ?

Solution: The temporal width at half of the peak power (FWHM) corresponds to the temporal width at $1/\sqrt{2}$ of the maximum of the Gaussian pulse (remember that the Gaussian pulse is the envelope of the field, and that the power is proportional to the square of the field.) For the given value of the temporal width at FWHM of the power ($\Delta t = 50\text{ps}$), we can calculate the time variance σ_t using the relation $\Delta t = 1.665\sigma_t$. This leads to $\sigma_t(0) = 30.03\,\text{ps}$. The time variance at the fiber output $\sigma_t(L)$ can be calculated from:

$$\sigma_t(L) = \sigma_t(0)\sqrt{1 + \left(\frac{L}{L_D}\right)^2},$$

where $L_D = \frac{\sigma_t^2(0)}{\left|\beta_c^{(2)}\right|}$, and $\beta_c^{(2)} = -\frac{\lambda^2}{2\pi c} C_\lambda$.

For both fibers with their respective length and dispersion it can therefore be calculated that $\sigma_t = 74.2\,\text{ps}$ and $\Delta t = 123.6\,\text{ps}$. The difference between both pulses is their chirp, which is of the same magnitude but opposite sign. That means that the spectral distribution of both pulses is reversed. For the pulse out of the SMF the "blue" (i.e., the shorter wavelength) components are at the leading edge of the pulse, whereas the "red" (i.e., the longer wavelength) components are at the trailing edge of the pulse.

c) Accidentally, DCF with $C_\lambda = -80\,\text{ps}/(\text{nm km})$ and a length as calculated in part a) were installed. How can this be compensated after five sections of SMF-DCF?

Solution: This can be solved by adding an SMF of length 26.25 km.

Exercise 2: Coupling to a low-index contrast waveguide

In this problem, we consider the power coupling efficiency between an integrated waveguide and a free-space beam. The waveguide is oriented along the $z-$direction. Assume that the waveguide has a low refractive index contrast such that the guided and radiative eigenmodes can be described by scalar functions $\underline{\Psi}_m(x,y)$ and $\underline{\Psi}_{\rho,\mu}(x,y)$ that represent the dominant E_x-components of the linearly polarized modes. The eigenmode expansion hence reads:

$$\Phi(x,y,z) = \sum_m a_m \underline{\Psi}_m(x,y)\,\mathrm{e}^{-\mathrm{j}\beta_m z} + \sum_\mu \int_\rho a_\mu(\rho)\,\underline{\Psi}_{\rho,\mu}(x,y) e^{-\mathrm{j}\beta_\mu(\rho) z}\,\mathrm{d}\rho.$$

All eigenmodes obey the simplified orthogonality relations:

$$\frac{\beta_\nu}{2\omega\mu_0}\iint_{-\infty}^{\infty}\underline{\Psi}_\nu(x,y)\,\underline{\Psi}_\mu^\star(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y=\mathcal{P}_\mu\delta_{\nu\mu}$$
$$\frac{\beta_{\rho,\nu}}{2\omega\mu_0}\iint_{-\infty}^{\infty}\underline{\Psi}_{\rho,\nu}(x,y)\,\underline{\Psi}_\mu^\star(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y=0,$$

where $\mathcal{P}_\mu$ is given by:

$$\mathcal{P}_\mu=\frac{\beta_\mu}{2\omega\mu_0}\iint_{-\infty}^{\infty}\left|\underline{\boldsymbol{\Psi}}_\mu(x,y)\right|^2\mathrm{d}\,x\,\mathrm{d}\,y.$$

To couple light into the waveguide, the facet (located at $z=0$) is illuminated with a linearly polarized monochromatic free-space beam. The E_x-component of the excitation is given by $\Phi(x,y,0)$. Assume that the waveguide has a low index of refraction such that the reflection at the facet surface can be neglected.

a) Derive an expression for the fundamental mode amplitude a_0.

Solution: To do this the mode amplitude a_0 has to be 'projected' out of the eigenmode expansion, which is done by multiplying the eigenmode expansion $\Phi(x,y,0)$ at the waveguide facet ($z=0$) by the complex conjugate of the transverse field distribution of the fundamental mode $\underline{\Psi}_0^*$ and subsequently integration over the transverse coordinates:

$$\iint\Phi(x,y,0)\underline{\Psi}_0^*\,\mathrm{d}\,x\,\mathrm{d}\,y=\sum_m a_m\underline{\Psi}_m(x,y)\Psi_0^*+\sum_\mu\int_\rho a_\mu(\rho)\,\underline{\Psi}_{\rho,\mu}(x,y)\Psi_0^*\,\mathrm{d}\,\rho.$$

Applying the orthogonality relations stated above, and solving for the mode amplitude a_0, the following relation can be derived:

$$a_0=\frac{\iint\Phi(x,y,0)\underline{\Psi}_0^*\,\mathrm{d}\,x\,\mathrm{d}\,y}{\iint|\underline{\Psi}_0|^2\,\mathrm{d}\,x\,\mathrm{d}\,y}.$$

b) What is the power carried by the fundamental mode? Note that the time-averaged power flux of the total E_x-polarized field distribution $\Phi(x,y,z)$ is given by:

$$P(z)=\frac{1}{2}\iint_{-\infty}^{\infty}\mathrm{Re}\left[\Phi(x,y,z)\,\frac{1}{\mathrm{j}\,\omega\mu_0}\frac{\partial\Phi^\star(x,y,z)}{\partial z}\right]\mathrm{d}\,x\,\mathrm{d}\,y.$$

Solution:

$$P_{\text{fund}}(0)=\frac{1}{2}\iint_{-\infty}^{\infty}\mathrm{Re}\left[a_0\underline{\Psi}_0\frac{\beta_0}{\omega\mu_0}a_0^*\underline{\Psi}_0^*\right]\mathrm{d}\,x\,\mathrm{d}\,y=$$
$$=\frac{\beta_0}{2\omega\mu_0}\iint_{-\infty}^{\infty}|a_0|^2\,|\underline{\Psi}_0|^2\,\mathrm{d}\,x\,\mathrm{d}\,y.$$

c) The fraction of power which is coupled from the incident field to the fundamental waveguide mode (mode index 0) is given by the power coupling efficiency η_0. Derive an expression for η_0 depending on the incident excitation field $\Phi(x,y,0)$ and the fundamental mode field $\underline{\Psi}_0(x,y)$. Assume that the power of the incident excitation field can be approximated by:

$$P(0)=\frac{\beta_0}{2\omega\mu_0}\iint_{-\infty}^{\infty}|\Phi(x,y,0)|^2\,\mathrm{d}\,x\,\mathrm{d}\,y.$$

Solution:

$$\eta_0=\frac{P_{\text{fund}}(0)}{P(0)}=\frac{\iint_{-\infty}^{\infty}|a_0|^2\,|\underline{\Psi}_0|^2\,\mathrm{d}\,x\,\mathrm{d}\,y}{\iint_{-\infty}^{\infty}|\Phi(x,y,0)|^2\,\mathrm{d}\,x\,\mathrm{d}\,y}=$$
$$=\frac{\iint_{-\infty}^{\infty}\left|\frac{\iint\Phi(x,y,0)\underline{\Psi}_0^*\,\mathrm{d}\,x\,\mathrm{d}\,y}{\iint|\underline{\Psi}_0|^2\,\mathrm{d}\,x\,\mathrm{d}\,y}\right|^2|\underline{\Psi}_0|^2\,\mathrm{d}\,x\,\mathrm{d}\,y}{\iint_{-\infty}^{\infty}|\Phi(x,y,0)|^2\,\mathrm{d}\,x\,\mathrm{d}\,y}=$$
$$=\frac{\left|\iint_{-\infty}^{\infty}\Phi(x,y,0)\,\underline{\Psi}_0^*\,\mathrm{d}\,x\,\mathrm{d}\,y\right|^2}{\iint_{-\infty}^{\infty}|\underline{\Psi}_0|^2\,\mathrm{d}\,x\,\mathrm{d}\,y\iint_{-\infty}^{\infty}|\Phi(x,y,0)|^2\,\mathrm{d}\,x\,\mathrm{d}\,y}.$$

d) Explain qualitatively what the incident field distribution should look like in order to obtain maximum coupling efficiency to the fundamental mode.

Solution: The integral in the numerator of the above equation has the form of an overlap integral of the incident field $\Phi(x, y, 0)$ with the fundamental waveguide mode $\underline{\Psi}_0$. This integral is the larger the more similar the fields are. Thus the best coupling efficiency is obtained, when the incident field resembles the fundamental waveguide mode.

Questions and Comments:

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